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ON THE CALCULATION OF THE SPECTRA OF BURNSIDE TAMBARA FUNCTORS

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ABSTRACT. For a finite group G, a Tambara functor on G is regarded as a G-bivariant analog of a commutative ring. In our previous article, we consider a G-bivariant analog of the ideal theory for Tambara functors. In this article, we will demonstrate calculations of spectra of Burnside Tambara functors, when $G=\mathbb{Z}/q\mathbb{Z}$.

1. INTRODUCTION AND PRELIMINARIES

A *Tambara functor* is firstly defined by Tambara [8] in the name 'TNR-functor', to treat the multiplicative transfers of Green functors. (For the definitions of Green and Mackey functors, see [1].) Later it is used by Brun [2] to describe the structure of Witt-Burnside rings.

For a finite group G, a Tambara functor is also regarded as a G-bivariant analog of a commutative ring, as seen in [9]. As such, for example a G-bivariant analog of the fraction ring was considered in [3], and a G-bivariant analog of the semigroup-ring construction was discussed in [5] and [6], with relation to the Dress construction [7].

In this analogy, we considered a G-bivariant analog of the ideal theory for Tambara functors in our previous article [4]. In this article, we will demonstrate calculations of spectra of Burnside Tambara functors, when $G=\mathbb{Z}/q\mathbb{Z}$ for some prime number q.

Throughout this article, the unit of a finite group G will be denoted by e. Abbreviately we denote the trivial subgroup of G by e, instead of $\{e\}$. $H\leq G$ means H is a subgroup of G. ${}_G set$ denotes the category of finite G-sets and G-equivariant maps. If $H\leq G$ and $g\in G$, then ${}^gH=gHg^{-1}$ denotes the conjugate ${}^gH=gHg^{-1}$.

A ring is assumed to be commutative, with an additive unit 0 and a multiplicative unit 1. A ring homomorphism preserves 0 and 1.

For any category $\mathscr{C}$ and any pair of objects X and Y in $\mathscr{C}$, the set of morphisms from X to Y in $\mathscr{C}$ is denoted by $\mathscr{C}(X,Y)$.

First we briefly recall the definition of a Tambara functor and its ideal.

Definition 1.1. ([8]) A *Tambara functor T on G* is a triplet $T=(T^*,T_+,T_\bullet)$ of two covariant functors

$$T_+\colon {}_G set\to Set,\quad T_\bullet\colon {}_G set\to Set$$

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and one contravariant functor

$$T^*\colon {}_G set \to Set$$

which satisfies the following. Here *Set* is the category of sets.

(1) $T^\alpha = (T^*, T_+)$ is a Mackey functor on G.

(2) $T^\mu = (T^*, T_\bullet)$ is a semi-Mackey functor on G.
Since T^α, T^μ are semi-Mackey functors, we have $T^*(X) = T_+(X) = T_\bullet(X)$ for each $X \in \mathrm{Ob}({}_G set)$. We denote this by $T(X)$.

(3) (Distributive law) If we are given an exponential diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & A \arrow[l, "p"'] \arrow[r, phantom, "exp"] & Z \arrow[l, "\lambda"'] \arrow[d, "\rho"] \\ Y & & B \arrow[ll, "q"] \end{tikzcd}$$

in ${}_G set$, then

$$\begin{tikzcd} T(X) \arrow[d, "T_\bullet(f)"'] & T(A) \arrow[l, "T_+(p)"'] \arrow[r, "T^*(\lambda)"] & T(Z) \arrow[d, "T_\bullet(\rho)"] \\ T(Y) & \circlearrowleft & T(B) \arrow[ll, "T_+(q)"] \end{tikzcd}$$

is commutative.

If $T = (T^*, T_+, T_\bullet)$ is a Tambara functor, then $T(X)$ becomes a ring for each $X \in \mathrm{Ob}({}_G set)$. For each $f \in {}_G set(X, Y)$,

- $T^*(f)\colon T(Y) \to T(X)$ is a ring homomorphism.
- $T_+(f)\colon T(X) \to T(Y)$ is an additive homomorphism.
- $T_\bullet(f)\colon T(X) \to T(Y)$ is a multiplicative homomorphism.

$T^*(f), T_+(f), T_\bullet(f)$ are often abbreviated to $f^*, f_+, f_\bullet$.

In this article, a *Tambara functor* always means a Tambara functor on some finite group G.

Example 1.2. If we define Ω by

$$\Omega(X) = K_0({}_G set/X)$$

for each $X \in \mathrm{Ob}({}_G set)$, where the right hand side is the Grothendieck ring of the category of finite G-sets over X, then Ω becomes a Tambara functor on G. This is called the *Burnside Tambara functor.* For each $f \in {}_G set(X, Y)$,

$$f_\bullet\colon \Omega(X) \to \Omega(Y)$$

is the one determined by

$$f_\bullet(A \xrightarrow{p} X) = (\Pi_f(A) \xrightarrow{\varpi} Y) \quad ({}^\forall(A \xrightarrow{p} X) \in \mathrm{Ob}({}_G set/X)),$$

where $\Pi_f(A)$ and ϖ is

$$\Pi_f(A) = \left\{ (y, \sigma) \;\middle|\; \begin{array}{c} y \in Y, \\ \sigma\colon f^{-1}(y) \to A \ \text{ a map of sets,} \\ p \circ \sigma = \mathrm{id}_{f^{-1}(y)} \end{array} \right\},$$

$$\varpi(y, \sigma) = y.$$

G acts on $\Pi_f(A)$ by $g \cdot (y, \sigma) = (gy, {}^g\sigma)$, where ${}^g\sigma$ is the map defined by

$$ {}^g\sigma(x) = g\sigma(g^{-1}x) \quad ({}^\forall x \in f^{-1}(gy)). $$

Definition 1.3. Let T be a Tambara functor. For each $f \in {}_G set(X, Y)$, define $f_! \colon T(X) \to T(Y)$ by

$$ f_!(x) = f_\bullet(x) - f_\bullet(0) $$

for any $x \in T(X)$.

Remark 1.4. ([4]) Let T be a Tambara functor. We have the following for any $f \in {}_G set(X, Y)$.

(1) $f_!$ satisfies $f_!(x) f_!(y) = f_!(xy)$ for any $x, y \in T(X)$.
(2) If f is surjective, then we have $f_! = f_\bullet$.
(3) If

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "\eta"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pull-back diagram, then $f'_! \xi^* = \eta^* f_!$ holds.

(4) If

$$\begin{tikzcd} X \arrow[d, "f"'] & A \arrow[l, "p"'] & Z \arrow[l, "\lambda"'] \arrow[d, "\rho"] \\ Y & exp & \Pi \arrow[ll, "\varpi"] \end{tikzcd}$$

is an exponential diagram, then $\varpi_+ \rho_! \lambda^* = f_! p_+$ holds.

Definition 1.5. ([4]) Let T be a Tambara functor. An *ideal* $\mathscr{I}$ of T is a family of ideals $\mathscr{I}(X) \subseteq T(X)$ (${}^\forall X \in \mathrm{Ob}({}_G set)$) satisfying

(i) $f^*(\mathscr{I}(Y)) \subseteq \mathscr{I}(X)$,
(ii) $f_+(\mathscr{I}(X)) \subseteq \mathscr{I}(Y)$,
(iii) $f_!(\mathscr{I}(X)) \subseteq \mathscr{I}(Y)$

for any $f \in {}_G set(X, Y)$. These conditions also imply

$$ \mathscr{I}(X_1 \amalg X_2) \cong \mathscr{I}(X_1) \times \mathscr{I}(X_2) $$

for any $X_1, X_2 \in \mathrm{Ob}({}_G set)$.

Obviously when G is trivial, this definition of an ideal agrees with the ordinary definition of an ideal of a commutative ring.

Remark 1.6. For any ideal $\mathscr{I} \subseteq T$, we have $\mathscr{I}(\emptyset) = T(\emptyset) = 0$.

Definition 1.7. ([4]) An ideal $\mathfrak{p} \subsetneq T$ is *prime* if for any transitive $X, Y \in \mathrm{Ob}({}_G set)$ and any $a \in T(X)$, $b \in T(Y)$,

$$ \langle a \rangle \langle b \rangle \subseteq \mathfrak{p} \quad \Rightarrow \quad a \in \mathfrak{p}(X) \quad \text{or} \quad b \in \mathfrak{p}(Y) $$

is satisfied. Remark that the converse always holds.

An ideal $\mathfrak{m} \subsetneq T$ is *maximal* if it is maximal with respect to the inclusion of ideals not equal to T. A maximal ideal is always prime.

Definition 1.8. ([4]) For any Tambara functor T on G, define $Spec(T)$ to be the set of all prime ideals of T. For each ideal $\mathscr{I} \subseteq T$, define a subset $V(\mathscr{I}) \subseteq Spec(T)$ by

$$V(\mathscr{I}) = \{\mathfrak{p} \in Spec(T) \mid \mathscr{I} \subseteq \mathfrak{p}\}.$$

Remark 1.9. ([4]) For any Tambara functor T, we have the following.

(1) $V(\mathscr{I}) = \emptyset$ if and only if $\mathscr{I} = T$.
(2) $V(\mathscr{I}) = Spec(T)$ if and only if $\mathscr{I} \subseteq \bigcap\limits_{\mathfrak{p} \in Spec(T)} \mathfrak{p}$.

Remark 1.10. ([4]) For any Tambara functor T, the family $\{V(\mathscr{I}) \mid \mathscr{I} \subseteq T \text{ is an ideal}\}$ forms a system of closed subsets of $Spec(T)$. Thus $Spec\,\Omega$ becomes a topological space.

2. Some propositions

Proposition 2.1. *Let T be a Tambara functor. Suppose we are given a family of ideals indexed by the set of finite non-empty transitive G-sets*

$$\{\mathscr{I}(X_0) \subseteq T(X_0)\}_{\substack{\text{transitive} \\ \emptyset \neq X_0 \in \mathrm{Ob}({}_G set)}}. \tag{2.1}$$

For any $X \in \mathrm{Ob}({}_G set)$, take its orbit decomposition $X = \coprod\limits_{1 \le i \le s} X_i$ and put

$$\mathscr{I}(X) = \mathscr{I}(X_1) \times \cdots \times \mathscr{I}(X_s) \subseteq T(X).$$

(We used the identification $T(X) \cong \prod\limits_{1 \le i \le s} T(X_i)$.) Then the following are equivalent.

(1) *$\mathscr{I} = \{\mathscr{I}(X)\}_{X \in \mathrm{Ob}({}_G set)}$ is an ideal of T.*
(2) *The family (2.1) satisfies*
   (i) $f^*(\mathscr{I}(Y_0)) \subseteq \mathscr{I}(X_0)$
   (ii) $f_+(\mathscr{I}(X_0)) \subseteq \mathscr{I}(Y_0)$
   (iii) $f_\bullet(\mathscr{I}(X_0)) \subseteq \mathscr{I}(Y_0)$
   for any transitive $X_0, Y_0 \in \mathrm{Ob}({}_G set)$ and any $f \in {}_G set(X_0, Y_0)$

Proof. Remark that for any non-empty transitive $X_0, Y_0 \in \mathrm{Ob}({}_G set)$ and any $f \in {}_G set(X_0, Y_0)$, we have $f_\bullet = f_!$. Obviously, (1) implies (2). We will show the converse.

Assume (2) holds. It suffices to show $\mathscr{I}$ satisfies (i), (ii), (iii) in Definition 1.5 for any $f \in {}_G set(X, Y)$.

First, we reduce to the case where Y is transitive. Take the orbit decomposition $Y = \coprod\limits_{1 \le j \le t} Y_j$, put

$$X_j = f^{-1}(Y_j), \quad f_j = f|_{X_j} : X_j \to Y_j,$$

and suppose (i), (ii), (iii) in Definition 1.5 holds for each f_j. Since we have commutative diagrams

$$\begin{tikzcd} T(X) \arrow[r, "\cong"] \arrow[d, "f_+"'] \arrow[dr, phantom, "\circlearrowleft"] & \prod_j T(X_j) \arrow[d, "\prod_j f_{j+}"] \\ T(Y) \arrow[r, "\cong"'] & \prod_j T(Y_j) \end{tikzcd}, \quad \begin{tikzcd} T(Y) \arrow[r, "\cong"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\circlearrowleft"] & \prod_j T(Y_j) \arrow[d, "\prod_j f_j^*"] \\ T(X) \arrow[r, "\cong"'] & \prod_j T(X_j) \end{tikzcd}, \quad \begin{tikzcd} T(X) \arrow[r, "\cong"] \arrow[d, "f_!"'] \arrow[dr, phantom, "\circlearrowleft"] & \prod_j T(X_j) \arrow[d, "\prod_j f_{j!}"] \\ T(Y) \arrow[r, "\cong"'] & \prod_j T(Y_j) \end{tikzcd},$$

under the canonical identification, we obtain

$$f_+(\mathscr{I}(X)) = \prod_i f_{j+}(\mathscr{I}(X_j)) \subseteq \prod_j \mathscr{I}(Y_j) = \mathscr{I}(Y),$$
$$f^*(\mathscr{I}(Y)) = \prod_i f_j^*(\mathscr{I}(Y_j)) \subseteq \prod_j \mathscr{I}(X_j) = \mathscr{I}(X),$$
$$f_!(\mathscr{I}(X)) = \prod_i f_{j!}(\mathscr{I}(X_j)) \subseteq \prod_j \mathscr{I}(Y_j) = \mathscr{I}(Y).$$

Now it remains to show in the case Y is transitive. If $X = \emptyset$, then there is nothing to show. Otherwise, take the orbit decomposition $X = \coprod_{1 \le i \le s} X_i$ and put $f_i = f|_{X_i} \colon X_i \to Y$. Remark that in this case, we have $f_\bullet = f_!$. By assumption, each f_i satisfies

$$\begin{aligned} f_{i+}(\mathscr{I}(X_i)) &\subseteq \mathscr{I}(Y), \\ f_i^*(\mathscr{I}(Y)) &\subseteq \mathscr{I}(X_i), \\ f_{i\bullet}(\mathscr{I}(X_i)) &\subseteq \mathscr{I}(Y). \end{aligned}$$

Under the identification $T(X) \cong \prod_{1 \le i \le s} T(X_i)$, we obtain $f^*(\mathscr{I}(Y)) \subseteq \mathscr{I}(X_1) \times \cdots \times \mathscr{I}(X_s) = \mathscr{I}(X)$. Moreover, for any $x \in \mathscr{I}(X)$, under the identification

$$\begin{aligned} \mathscr{I}(X) &= \mathscr{I}(X_1) \times \cdots \mathscr{I}(X_s) \\ x &= (x_1, \ldots, x_s), \end{aligned}$$

we have

$$\begin{aligned} f_+(x) &= f_{1+}(x_1) + \cdots + f_{s+}(x_s) \in \mathscr{I}(Y), \\ f_\bullet(x) &= f_{1\bullet}(x_1) \cdot \cdots \cdot f_{s\bullet}(x_s) \in \mathscr{I}(Y). \end{aligned}$$

Thus it follows $f_+(\mathscr{I}(X)) \subseteq \mathscr{I}(Y)$, $f_\bullet(\mathscr{I}(X)) \subseteq \mathscr{I}(Y)$. □

Corollary 2.2. *To give an ideal $\mathscr{I}$ of a Tambara functor T on G is equivalent to give a family of ideals indexed by $\mathcal{O}_G$*

$$\{\mathscr{I}(G/H) \subseteq T(G/H)\}_{H \in \mathcal{O}(G)}$$

satisfying

(i) $\mathrm{res}^H_K(\mathscr{I}(G/H)) \subseteq \mathscr{I}(G/K)$
(ii) $\mathrm{ind}^H_K(\mathscr{I}(G/K)) \subseteq \mathscr{I}(G/H)$
(iii) $\mathrm{jnd}^H_K(\mathscr{I}(G/K)) \subseteq \mathscr{I}(G/H)$
(iv) $c_{g,H}(\mathscr{I}(G/H)) \subseteq \mathscr{I}(G/{}^gH)$

for any $K \le H \le G$ and $g \in G$. In particular, $\mathscr{I}(G/H) \subseteq T(G/H)$ is $N_G(H)/H$-invariant.

By construction, for ideals $\mathscr{I}, \mathscr{J} \subseteq T$, we have

$$\mathscr{I} \subseteq \mathscr{J} \Leftrightarrow \mathscr{I}(G/H) \subset \mathscr{J}(G/H) \ ({}^\forall H \in \mathcal{O}(G)).$$

Corollary 2.3. *When $G = \mathbb{Z}/q\mathbb{Z}$ where q is a prime number, then to give an ideal $\mathscr{I}$ of T is equivalent to give*

- *a G-invariant ideal $\mathscr{I}(G/e) \subseteq T(G/e)$,*
- *an ideal $\mathscr{I}(G/G) \subseteq T(G/G)$,*

satisfying

(i) $\pi^*(\mathscr{I}(G/G)) \subseteq \mathscr{I}(G/e)$,
(ii) $\pi_+(\mathscr{I}(G/e)) \subseteq \mathscr{I}(G/G)$,
(iii) $\pi_\bullet(\mathscr{I}(G/e)) \subseteq \mathscr{I}(G/G)$,

where $\pi\colon G/e \to G/G$ *is the unique constant map.*

Remark 2.4. (Corollary 4.5 in [4]) An ideal $\mathscr{I} \subseteq T$ is prime if and only if for any transitive $X, Y \in \mathrm{Ob}({}_G set)$ and any $a \in T(X), b \in T(Y)$, the following two conditions become equivalent.

(1) $a \in T(X)$ or $b \in T(Y)$.
(2) For any $C \in \mathrm{Ob}({}_G set)$ and for any pair of diagrams in ${}_G set$

$$C \xleftarrow{v} D \xrightarrow{w} X, \quad C \xleftarrow{v'} D' \xrightarrow{w'} Y,$$

$(v_! w^*(a)) \cdot (v'_! w'^*(b)) \in \mathscr{I}(C)$ is satisfied.

Note that (1) always implies (2).

By the following lemma, it is enough to check (2) only when C, D, D' are transitive.

Lemma 2.5. *Let* $\mathscr{I} \subseteq T$ *be an ideal. Condition* (2) *in Remark 2.4 is equivalent to the following.*

(2)′ *For any* transitive $C \in \mathrm{Ob}({}_G set)$ *and for any pair of diagrams in* ${}_G set$

$$C \xleftarrow{v} D \xrightarrow{w} X, \quad C \xleftarrow{v'} D' \xrightarrow{w'} Y$$

where D *and* D' *are* transitive, $(v_\bullet w^*(a)) \cdot (v'_\bullet w'^*(b)) \in \mathscr{I}(C)$ *is satisfied.*

Proof. It suffices to show (2)′ implies (2). Assume (2)′ holds, take any $C \in \mathrm{Ob}({}_G set)$ and

$$C \xleftarrow{v} D \xrightarrow{w} X, \quad C \xleftarrow{v'} D' \xrightarrow{w'} Y,$$

with not necessarily transitive C, D, D'.

Let $C = \coprod_{a \leq i \leq m} C_i$ be the orbit decomposition, and put

$$\begin{aligned} D_i = v^{-1}(C_i) \quad &, \quad D'_i = v'^{-1}(C_i), \\ v_i = v|_{D_i}\colon D_i \to C_i \quad &, \quad v'_i = v'|_{D'_i}\colon D'_i \to C_i, \\ w_i = w|_{D_i}\colon D_i \to X \quad &, \quad w'_i = w'|_{D'_i}\colon D'_i \to Y. \end{aligned}$$

Then we have $v_! w^*(a) = (v_{1!} w_1^*(a), \ldots, v_{m!} w_m^*(a))$, where

$$v_{i!} w_i^*(a) = \begin{cases} v_{i\bullet} w_i^*(a) & \text{if } D_i \neq \emptyset \\ 0 & \text{if } D_i = \emptyset. \end{cases}$$

Similarly for b. In any case, $(v_{i!} w_i^*(a)) \cdot (v'_{i!} w'^*_i(b)) \in \mathscr{I}(C_i)$ $(1 \leq {}^\forall i \leq m)$ follows from (2)′, which means

$$(v_! w^*(a)) \cdot (v'_! w'^*(b)) \in \mathscr{I}(C).$$

□

Proposition 2.6. *Let T be a Tambara functor, and $\mathfrak{p} \subseteq T$ be a prime ideal. Let $T(G/e)^G$ denote the subring of G-invariant elements in $T(G/e)$:*

$$T(G/e)^G = \{x \in T(G/e) \mid gx = x \ ({}^\forall g \in G)\}$$

Similarly for $\mathfrak{p}(G/e)^G$:

$$\mathfrak{p}(G/e)^G = \mathfrak{p}(G/e) \cap T(G/e)^G$$

Then, $\mathfrak{p}(G/e)^G \subseteq T(G/e)^G$ is a prime ideal (in the ordinary ring-theoretic meaning).

Proof. Suppose $a, b \in T(G/e)^G$ satisfies $ab \in \mathfrak{p}(G/e)$. By Lemma 2.5, it suffices to show for any transitive C, D, D' and any pair of diagrams in ${}_G set$

$$C \xleftarrow{v} D \xrightarrow{w} G/e, \quad C \xleftarrow{v'} D' \xrightarrow{w'} G/e, \tag{2.2}$$

$(v_\bullet w^*(a)) \cdot (v'_\bullet w'^*(b)) \ \in \mathfrak{p}(C)$ is satisfied. Since D and D' are transitive with trivial stabilizers, we may assume $D = D' = G/e$. Furthermore, modifying v and v' by conjugations, we may assume

$$C = G/H, \quad v = v' = p_e^H : G/H \to G/e$$

for some $H \leq G$. Thus (2.2) is reduced to the case

$$G/H \xleftarrow{p_e^H} G/e \xrightarrow{w} G/e, \quad G/H \xleftarrow{p_e^H} G/e \xrightarrow{w'} G/e,$$

where w, w' are the multiplication by some $g, g' \in G$. Then we have

$$\begin{aligned} (\,(p_e^H)_\bullet w^*(a)\,) \cdot (\,(p_e^H)_\bullet w'^*(b)\,) &= (p_e^H)_\bullet((ga) \cdot (g'b)) \\ &= (p_e^H)_\bullet(ab) \ \in \mathfrak{p}(G/H). \end{aligned}$$

□

Corollary 2.7. *If $\mathfrak{p} \subseteq \Omega$ is prime, then $\mathfrak{p}(G/e) \subseteq \Omega(G/e)$ is prime.*

Proof. This immediately follows from the fact that $\Omega(G/e) \cong \mathbb{Z}$ has a trivial G-action. □

3. *Spec* Ω for $G = \mathbb{Z}/q\mathbb{Z}$

In the following, we assume $G = \mathbb{Z}/q\mathbb{Z}$ for some prime number q, and denote the canonical projection by $\pi = p_e^G : G/e \to G/G$.

3.1. Structure of Ω.

Proposition 3.1. *For $G = \mathbb{Z}/q\mathbb{Z}$, Burnside Tambara functor has the following structure.*

(1) *There are isomorphisms of rings*

$$\begin{aligned} \Omega(G/e) &\xrightarrow{\cong} \mathbb{Z} \ ; \ \ell G/e \mapsto \ell, \\ \Omega(G/G) &\xrightarrow{\cong} \mathbb{Z}[X]/(X^2 - qX) \ ; \ mG/e + nG/G \mapsto m + nX. \end{aligned}$$

(2) *Under the isomorphisms in (1), the structure morphisms $\pi_+, \pi^*, \pi_\bullet$ are*

$$\begin{aligned} \pi_+ &: \mathbb{Z} \to \mathbb{Z}[X]/(X^2 - qX) \ ; \ \ell \mapsto \ell X, \\ \pi^* &: \mathbb{Z}[X]/(X^2 - qX) \to \mathbb{Z} \ ; \ m + nX \mapsto m + qn, \\ \pi_\bullet &: \mathbb{Z} \to \mathbb{Z}[X]/(X^2 - qX) \ ; \ \ell \mapsto \ell + \frac{\ell^q - \ell}{q} X. \end{aligned}$$

Proof. The only non-trivial part will be

$$\pi_\bullet(\ell) = \ell + \frac{\ell^q - \ell}{q} X.$$

This is shown by using the following.

Fact 3.2. (Proposition 4.17 in [4])
The following diagram is commutative.

$$\begin{tikzcd}
\ell \arrow[r, mapsto] & \ell & \\
\Omega(G/e) \arrow[r] \arrow[d, "\pi_\bullet"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z} & m \\
\Omega(G/G) \arrow[ur] & m+nX \arrow[ur, mapsto] &
\end{tikzcd}$$

From this fact, for any $\ell \in \mathbb{Z}$ we have

(3.1) $$\pi_\bullet(\ell) = \ell + nX$$

for some $n \in \mathbb{Z}$. Remark that $n \geq 0$ holds if $\ell \geq 0$.

Besides, by the definition of $\pi_\bullet$, for any $\ell \in \mathbb{N}_{\geq 0}$ we have

$$\pi_\bullet(\coprod_\ell G/e \xrightarrow{\nabla} G/e) = \{\sigma \mid \sigma \colon G/e \to \coprod_\ell G/e, \text{ a section map for } \nabla\},$$

and thus

(3.2) $$\sharp(\pi_\bullet(\ell)) = \ell^q.$$

From (3.1) and (3.2),

$$\pi_\bullet(\ell) = \ell + \frac{\ell^q - \ell}{q} X$$

for any $\ell \geq 0$. As for a negative ℓ, since we have

$$\pi_\bullet(\ell) = \pi_\bullet(-1)\pi_\bullet(|\ell|),$$

it will be enough to determine $\pi_\bullet(-1)$.

By (3.1), we have $\pi_\bullet(-1) = -1 + nX$ for some $n \in \mathbb{Z}$, which should satisfy

$$1 = \pi_\bullet(-1)^2 = (-1 + nX)^2 = 1 + n(qn - 2)X.$$

When q is odd, it follows $n = 0$, and $\pi_\bullet(-1) = -1$. For $q = 2$, both -1 and $-1 + X$ satisfy $(-1)^2 = (-1 + X)^2 = 1$. However, from the Mackey condition for the pullback

$$\begin{tikzcd}
\coprod_2 G/e \arrow[r, "\nabla"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\square"] & G/e \arrow[d, "\pi"] \\
G/e \arrow[r, "\pi"'] & G/G
\end{tikzcd},$$

$\pi_\bullet(-1)$ should satisfy

$$\pi^* \pi_\bullet(-1) = 1,$$

which leads to $\pi_\bullet(-1) = -1 + X$.

In any case, we obtain

$$\pi_\bullet(\ell) = \ell + \frac{\ell^q - \ell}{q} X \quad ({}^\forall \ell \in \mathbb{Z})$$

for any prime q. $\square$

3.2. **Decomposition into fibers.** Using the structural isomorphism in Proposition 3.1, we go on to determine $Spec\,\Omega$ for $G = \mathbb{Z}/q\mathbb{Z}$. By Corollary 2.7, any prime ideal $\mathfrak{p} \subseteq \Omega$ satisfies $\mathfrak{p}(G/e) = (p)$ for some prime p or $p = 0$. Thus we have a map

$$F\colon Spec\,\Omega \to Spec\,\mathbb{Z} \ \ ; \ \ \mathfrak{p} \mapsto \mathfrak{p}(G/e).$$

(F will be shown to be continuous after $Spec\,\Omega$ is determined.)

Definition 3.3. Let $p \in \mathbb{Z}$ be prime or $p = 0$. We call an ideal $\mathscr{I} \subseteq \Omega$ *is over* p if it satisfies $\mathscr{I}(G/e) = (p)$. A *prime ideal over* p is simply a prime ideal $\mathfrak{p} \subseteq \Omega$ which is over p.

Remark 3.4. By the above arguments, we have

- $F^{-1}((p)) = \{\mathfrak{p} \in Spec\,\Omega \mid \text{prime ideal over } p\}$,
- $Spec\,\Omega = \coprod_{(p)\in Spec\,\mathbb{Z}} F^{-1}((p))$.

In the following, we investigate the fibers $F^{-1}((p))$, in the cases $p = 0$, $p = q$, and $p \neq 0, q$.

For each $(p) \in Spec\,\mathbb{Z}$, its fiber $F^{-1}((p))$ at least contains one maximal point. In fact, the following was shown in [4].

Fact 3.5. (Corollary 4.42 in [4])

$$Spec\,\Omega \supseteq \{\mathscr{I}_{(p)} \mid p \in \mathbb{Z} \text{ is prime}\} \cup \{\mathscr{I}_{(0)}\} \cup \{(0)\}.$$

Here, for each ideal $I \subseteq \Omega(G/e)$, ideal $\mathscr{I}_I \subseteq \Omega$ is defined by

$$\mathscr{I}_I(G/e) = I, \quad \mathscr{I}_I(G/G) = (\pi^*)^{-1}(I).$$

$\mathscr{I}_I$ is the largest one, among all ideals $\mathscr{I} \subseteq \Omega$ satisfying $\mathscr{I}(G/e) = I$.

Under the isomorphism in Proposition 3.1, for any $\ell \in \mathbb{Z}$ we have

$$\begin{aligned}
\mathscr{I}_{(\ell)}(G/e) &= (\ell) \ \subseteq \mathbb{Z}, \\
\mathscr{I}_{(\ell)}(G/G) &= \{m + nX \in \mathbb{Z}[X]/(X^2 - qX) \mid m + qn \in (\ell)\} \\
&= \{k\ell + n(X - q) \in \mathbb{Z}[X]/(X^2 - qX) \mid k, n \in \mathbb{Z}\} \\
&= (\ell, X - q) \ \subseteq \mathbb{Z}[X]/(X^2 - qX).
\end{aligned}$$

In this article, we denote $\mathscr{I}_{(p)}$ by $\mathfrak{m}_p$. For any prime $p \neq 0$, $\mathfrak{m}_p$ is a maximal ideal of Ω. Namely it is a closed point in $Spec\,\Omega$, while $\mathfrak{m}_0 = \mathscr{I}_{(0)}$ is not. (For this reason, we prefer to use $\mathscr{I}_{(0)}$ rather than $\mathfrak{m}_0$ only for $p = 0$.)

On the other hand, (0) is the smallest ideal of Ω, namely the generic point in $Spec\,\Omega$. We have inclusions

$$(0) \subsetneq \mathscr{I}_{(0)} \subsetneq \mathfrak{m}_p$$

for any prime $p \in \mathbb{Z}$.

3.3. **The smallest ideal over p.**

Proposition 3.6. *For a prime $p \in \mathbb{Z}$ or $p = 0$, the smallest ideal $I_p \subseteq \Omega$ over p is given by the following.*

(1) *When $p \neq q$ (including the case $p = 0$),*

$$I_p(G/G) = (p) \ \subseteq \mathbb{Z}[X]/(X^2 - qX).$$

(2) *When $p = q$,*

$$I_q(G/G) = (qX, X - q) = (q^2, X - q) \ \subseteq \mathbb{Z}[X]/(X^2 - qX).$$

Proof. (1) $(p) \subseteq I_p(G/e)$ follows from

$$\begin{aligned} p &= (p+\frac{p^q-p}{q}X) - \frac{p^q-p}{pq}\cdot pX \\ &= \pi_\bullet(p) - \frac{p^q-p}{pq}\pi_+(p). \end{aligned}$$

To show the converse, it suffices to show that

$$\mathscr{I}(G/e) = (p) \subseteq \mathbb{Z} \text{ and } \mathscr{I}(G/G) = (p) \subseteq \mathbb{Z}[X]/(X^2-qX)$$

in fact form an ideal $\mathscr{I}$ of Ω. By Corollary 2.3, this is equivalent to show

$$\begin{aligned} \pi^*((p)) &\subseteq (p), \\ \pi_+((p)) &\subseteq (p), \\ \pi_\bullet((p)) &\subseteq (p). \end{aligned}$$

However, these immediately follow from

$$\pi^*(p) = p \quad \in (p)$$

and

$$\begin{aligned} \pi_+(\ell p) &= \ell pX \quad \in (p) \\ \pi_\bullet(\ell p) &= \ell p + \frac{\ell^q p^q - \ell p}{q}X \quad \in (p) \end{aligned}$$

for any $\ell \in \mathbb{Z}$. (Remark that π^* is a ring homomorphism.)

(2) $(qX, X-q) \subseteq I_q(G/e)$ follows from

$$qX = \pi_+(q)$$

and

$$X - q = q^{q-1}X - (q + \frac{q^q-q}{q}X) = \pi_+(q^{q-1}) - \pi_\bullet(q).$$

To show the converse, it suffices to show

$$\begin{aligned} \pi^*((q^2, X-q)) &\subseteq (q), \\ \pi_+((q)) &\subseteq (qX, X-q), \\ \pi_\bullet((q)) &\subseteq (qX, X-q). \end{aligned}$$

These follow from

$$\pi^*(q^2) = q^2, \quad \pi^*(X-q) = 0 \qquad \in (q),$$

and

$$\begin{aligned} \pi_+(\ell q) &= \ell qX \quad \in (qX) \\ \pi_\bullet(\ell q) &= \ell(q-X) + \ell^q q^{q-1}X \quad \in (q-X, qX) \end{aligned}$$

for any $\ell \in \mathbb{Z}$. □

3.4. All ideals over p.

For $p \neq 0$, ideals $\mathscr{I} \subseteq \Omega$ over p are only I_p and $\mathfrak{m}_p$.

Claim 3.7. *When $p \in \mathbb{Z}$ is prime ($\neq 0$), then there is no ideal between $I_p \subsetneq \mathfrak{m}_p$.*

Proof. It suffices to show that there is no element $f \in \Omega(G/G)$ satisfying

$$I_p(G/G) \subsetneq I_p(G/G) + (f) \subsetneq (p, X - q). \tag{3.3}$$

By $f \in (p, X - q)$, it should be of the form $f = kp + n(X - q)$ for some $k, n \in \mathbb{Z}$.

(1) When $p \neq q$, (3.3) is equal to

$$(p) \subsetneq (p, f) \subsetneq (p, X - q).$$

This will mean the existence of $n \in \mathbb{Z}$ satisfying $(p) \subsetneq (p, n(X - q)) \subsetneq (p, X - q)$. However, since

$$(p, n(X - q)) = \begin{cases} (p) & \text{if } \ p|n \\ (p, X - q) & \text{if } \ p\not| n \end{cases},$$

there should not exist such n.

(2) When $p = q$, (3.3) is equal to

$$(q^2, X - q) \subsetneq (q^2, X - q, f) \subsetneq (q, X - q).$$

This will mean the existence of $k \in \mathbb{Z}$ satisfying

$$(q^2, X - q) \subsetneq (q^2, X - q, kq) \subsetneq (q, X - q).$$

However, since

$$(q^2, X - q, kq) = \begin{cases} (q^2, X - q) & \text{if } \ q|k \\ (q, X - q) & \text{if } \ q\not| k \end{cases},$$

there should not exist such k. □

On the other hand for $p = 0$, there are many ideals between $(0) \subsetneq \mathscr{I}_{(0)}$.

Claim 3.8. *If we define $\mathscr{I}_{(0;n)} \subseteq \Omega$ by*

$$\mathscr{I}_{(0;n)}(G/e) = (0) \ , \quad \mathscr{I}_{(0;n)}(G/G) = n(X - q),$$

then $\mathscr{I}_{(0;n)} \subseteq \Omega$ forms an ideal for each $n \in \mathbb{Z}$. Indeed, these are exactly the all ideals $\mathscr{I} \subseteq \Omega$ over 0:

$$\{\mathscr{I} \subseteq \Omega \ \textit{ideal} \mid \mathscr{I}(G/e) = 0\} \quad = \quad \{\mathscr{I}_{(0;n)} \mid n \in \mathbb{Z}\}$$

Proof. Any ideal between $(0) \subsetneq (X-q)$ in $\mathbb{Z}[X]/(X^2-qX)$ is of the form $(n(X-q))$ for some $n \in \mathbb{Z}$. Since $\mathscr{I}_{(0;n)}(G/e) = (0)$ and $\mathscr{I}_{(0;n)}(G/G) = (n(X - q))$ satisfy

$$\pi^*(n(X - q)) = 0, \quad \pi_+(0) = 0, \quad \pi_\bullet(0) = 0,$$

$\mathscr{I}_{(0;n)} \subseteq \Omega$ gives an ideal for each $n \in \mathbb{Z}$. □

3.5. **Criterion to be prime.** Let $p \in \mathbb{Z}$ be a prime or $p = 0$. Now we give a criterion for an ideal $\mathscr{I} \subseteq \Omega$ over p to be prime.

Proposition 3.9. *Let $p \in \mathbb{Z}$ be a prime or $p = 0$. Let $\mathscr{I} \subseteq \Omega$ be an ideal over p, not equal to $\mathfrak{m}_p$. Then $\mathscr{I}$ is* not *prime if and only if one of the following conditions is satisfied.*

(c1) *There exist $a, b \in \mathfrak{m}_p(G/G)$ satisfying*

$$a \notin \mathscr{I}(G/G), \quad b \notin \mathscr{I}(G/G), \quad ab \in \mathscr{I}(G/G).$$

(c2) *There exist $a \in \mathfrak{m}_p(G/G)$ and $b \in \Omega(G/e)$ satisfying*

$$a \notin \mathscr{I}(G/G), \quad \pi_\bullet(b) \notin \mathscr{I}(G/G), \quad a \cdot (\pi_\bullet(b)) \in \mathscr{I}(G/G).$$

(Only here, we use the notation $\mathfrak{m}_0 = \mathscr{I}_{(0)}$ for the consistency.) In particular, if $\mathscr{I}(G/G) \subseteq \Omega(G/G)$ is prime, then $\mathscr{I} \subseteq \Omega$ is prime.

More explicitly, these can be written as follows.

(c1)′ *There exist $k, n, k', n' \in \mathbb{Z}$ satisfying*

$$kp + n(X - q) \notin \mathscr{I}(G/G), \quad k'p + n'(X - q) \notin \mathscr{I}(G/G),$$
$$kk'p^2 + ((n'k + nk')p + nn'q)(X - q) \in \mathscr{I}(G/G).$$

(c2)′ *There exist $k, n, l \in \mathbb{Z}$ satisfying*

$$kp + n(X - q) \notin \mathscr{I}(G/G), \quad \ell + \tfrac{\ell^q - \ell}{q} X \notin \mathscr{I}(G/G),$$
$$kp(\ell + \tfrac{\ell^q - \ell}{q} X) + n\ell(X - q) \in \mathscr{I}(G/G).$$

Proof. By Lemma 2.5, $\mathscr{I} \subseteq \Omega$ is not prime if and only if there exist transitive $X, Y \in \mathrm{Ob}({}_G set)$ and $a \in \Omega(X), b \in \Omega(Y)$ satisfying $a \notin \mathscr{I}(X), b \notin \mathscr{I}(Y)$ and

($\diamond$) $\quad (v_\bullet w^*(a)) \cdot (v'_\bullet w'^*(b)) \in \mathscr{I}(C)$ for any

$$C \xleftarrow{v} D \xrightarrow{w} X, \quad C \xleftarrow{v'} D' \xrightarrow{w'} Y,$$

with C, D, D' transitive.

We may consider this condition in the following three cases.

(1) $X = Y = G/e$.
(2) $X = Y = G/G$.
(3) $X = G/G$, $Y = G/e$.

(1) If $X = Y = G/e$, then ($\diamond$) is reduced to

$$ab \in \mathscr{I}(G/e) = (p),$$

which implies automatically a or b is in $\mathscr{I}(G/e)$. Thus we can exclude this case.

(2) If $X = Y = G/G$, then condition ($\diamond$) is equivalent to

$$ab \in \mathscr{I}(G/G), \quad \pi^*(a)\pi^*(b) \in \mathscr{I}(G/G),$$
$$(\pi_\bullet\pi^*(a)) \cdot b \in \mathscr{I}(G/G), \quad a \cdot (\pi_\bullet\pi^*(b)) \in \mathscr{I}(G/G),$$
$$(\pi_\bullet\pi^*(a)) \cdot (\pi_\bullet\pi^*(b)) \in \mathscr{I}(G/G.).$$

Since $\mathscr{I}(G/e) = (p)$ is prime, it follows that $\pi^*(a)$ or $\pi^*(b)$ is in $\mathscr{I}(G/e)$. Thus we may assume $\pi^*(a) \in (p)$, namely $a \in \mathfrak{m}_p(G/G)$. Then the above conditions are reduced to

$$ab \in \mathscr{I}(G/G), \quad a \cdot (\pi_\bullet\pi^*(b)) \in \mathscr{I}(G/G).$$

The existence of such a and b can be divided into the following two cases. Remark that $\pi^*(b) \notin \mathscr{I}(G/e)$ will imply $b \notin \mathscr{I}(G/G)$.

(2-1) (the case $\pi^*(b) \notin (p)$)
There exist $a \in \mathfrak{m}_p(G/G)$ and $b \in \Omega(G/G)$ satisfying

$$a \notin \mathscr{I}(G/G), \quad \pi^*(b) \notin \mathscr{I}(G/e),$$
$$ab \in \mathscr{I}(G/G), \quad a \cdot (\pi_\bullet \pi^*(b)) \in \mathscr{I}(G/G).$$

(2-2) (the case $\pi^*(b) \in (p)$)
There exist $a, b \in \mathfrak{m}_p(G/G)$ satisfying

$$a \notin \mathscr{I}(G/G), \quad b \notin \mathscr{I}(G/G), \quad ab \in \mathscr{I}(G/G).$$

(3) If $X = G/G$ and $Y = G/e$, then for $a \in \Omega(G/G)$ and $b \in \Omega(G/e)$ which are not in $\mathscr{I}$, condition ($\diamond$) is reduced to

$$(\pi^*(a)) \cdot b \in \mathscr{I}(G/e), \quad a \cdot (\pi_\bullet(b)) \in \mathscr{I}(G/G).$$

Since $b \notin \mathscr{I}(G/e) = (p)$, the condition $(\pi^*(a)) \cdot b \in \mathscr{I}(G/e)$ is equivalent to $\pi^*(a) \in \mathscr{I}(G/e)$, namely to $a \in \mathfrak{m}_p(G/G)$. The existence of such a and b can be divided into the following two cases. Remark that $\pi_\bullet(b) \notin \mathscr{I}(G/G)$ will imply $b \notin \mathscr{I}(G/e)$.

(3-1) (the case $\pi_\bullet(b) \notin \mathscr{I}(G/G)$)
There exist $a \in \mathfrak{m}_p(G/G)$ and $b \in \Omega(G/e)$ satisfying

$$a \notin \mathscr{I}(G/G), \quad \pi_\bullet(b) \notin \mathscr{I}(G/G), \quad a \cdot (\pi_\bullet(b)) \in \mathscr{I}(G/G).$$

(3-2) (the case $\pi_\bullet(b) \in \mathscr{I}(G/G)$)
There exist $a \in \mathfrak{m}_p(G/G)$ and $b \in \Omega(G/e)$ satisfying

$$a \notin \mathscr{I}(G/G), \quad b \notin \mathscr{I}(G/e), \quad \pi_\bullet(b) \in \mathscr{I}(G/G).$$

Note that, in (3-2), the conditions for a and b are completely separated. Moreover since $\mathscr{I}(G/G) \subsetneq \mathfrak{m}_p(G/G)$, such a always exists. Thus (3-2) is reduced to the following.

(3-2)′ There exists $b \in \Omega(G/e)$ satisfying

$$b \notin \mathscr{I}(G/e) \ \text{ and } \ \pi_\bullet(b) \in \mathscr{I}(G/G).$$

However, this never happens. Indeed, since we have

$$\pi^*\pi_\bullet(\ell) = \ell^q$$

for any $\ell \in \Omega(G/e)$, we obtain

$$\pi_\bullet(\ell) \Rightarrow \pi^*\pi_\bullet(b) \in \mathscr{I}(G/e) \Rightarrow \ell \in \mathscr{I}(G/e).$$

By the arguments so far, $\mathscr{I} \subseteq \Omega$ is not prime if and only if one of (2-1), (2-2), (3-1) is satisfied. Furthermore, we see (2-1) implies (3). Indeed if a and b satisfy (2-1), then $a \in \Omega(G/G)$ and $b' = \pi^*(b) \in \Omega(G/e)$ satisfy

$$a \notin \mathscr{I}(G/G), \quad b' \notin \mathscr{I}(G/e),$$
$$a \cdot (\pi_\bullet(b')) \in \mathscr{I}(G/G), \quad \pi^*(a) \cdot b' = \pi^*(ab) \in \mathscr{I}(G/e).$$

Thus, we can conclude that $\mathscr{I} \subseteq \Omega$ is not prime if and only if one of (2-2), (3-1) is satisfied. These are respectively the conditions (c1), (c2) in the statement of the proposition.

The latter part can be shown easily by using $\mathfrak{m}_p(G/G) = (p, X-q)$. An easy observation $X(X-q) = 0$ will help the calculation. $\square$

3.6. **Determine each fiber.** Proposition 3.9 enables us to determine the structure of $Spec\,\Omega$.

Corollary 3.10. *Let $p \in \mathbb{Z}$ be a prime or $p = 0$. In each fiber $F^{-1}((p))$ over p, we have the following.*

(1) (*the case $p \neq q, 0$*)
If $p \neq 0$ is a prime other than q, then $I_p \subseteq \Omega$ in Proposition 3.9 is prime. For this reason, in the rest we denote I_p by $\mathfrak{p}_p$.

(2) (*the case $p = q$*)
$I_q \subseteq \Omega$ is not prime.

(3) (*the case $p = 0$*)
$\mathscr{I}_{(0;n)} \subseteq \Omega$ in Claim 3.8 is prime if and only if $n = 0$ or $n = \pm 1$.

Proof. (1) It suffices to show that either of (c1)$'$, (c2)$'$ does not occur. Remark that we have $\mathfrak{p}_p(G/G) = (p)$.

(c1)$'$ For any k, n, k', n', since

$$\begin{aligned} kp + n(X-q) \notin \mathfrak{p}_p(G/G) \quad &\Leftrightarrow \quad p \nmid n, \\ k'p + n'(X-q) \notin \mathfrak{p}_p(G/G) \quad &\Leftrightarrow \quad p \nmid n', \\ kk'p^2 + ((n'k + nk')p + nn'q)(X-q) \in \mathfrak{p}_p(G/G) \quad &\Leftrightarrow \quad p \mid nn', \end{aligned}$$

these never happens simultaneously.

(c2)$'$ For any $k, n, l \in \mathbb{Z}$, since

$$\begin{aligned} kp + n(X-q) \notin \mathfrak{p}_p(G/G) \quad &\Leftrightarrow \quad p \nmid n, \\ \ell + \frac{\ell^q - \ell}{q} X \notin \mathfrak{p}_p(G/G) \quad &\Leftrightarrow \quad p \nmid \ell, \\ kp(\ell + \frac{\ell^q - \ell}{q} X) + n\ell(X-q) \in \mathfrak{p}_p(G/G) \quad &\Leftrightarrow \quad p \mid n\ell, \end{aligned}$$

these never happens simultaneously.

(2) We show (c1) holds for I_q. Remark that we have $I_q(G/G) = (qX, X-q)$.
For $a = b = X \in \mathfrak{m}_q(G/G)$, we have

$$a = b \notin I_q(G/G) \quad \text{and} \quad ab = qX \in I_q(G/G).$$

Thus I_q is not prime.

(3) We already know $(0) \subseteq \Omega$ and $\mathscr{I}_{(0)} \subseteq \Omega$ are prime. It suffices to show $\mathscr{I}_{(0;n)} \subseteq \Omega$ is not prime for $n \notin \{-1, 0, 1\}$. We show (c2) holds for these n. Remark that we have $\mathscr{I}_{(0;n)}(G/G) = (n(X-q))$.
For $a = X - q \in \Omega(G/G)$ and $b = n \in \Omega(G/e)$, we have

$$\begin{gathered} a \notin \mathscr{I}_{(0;n)}(G/G), \\ \pi_\bullet(b) = n + \tfrac{n^q - n}{q} X \notin \mathscr{I}_{(0;n)}(G/G), \\ (X-q) \cdot (\pi_\bullet(b)) = n(X-q) \in \mathscr{I}_{(0;n)}(G/G). \end{gathered}$$

Thus $\mathscr{I}_{(0;n)}$ is not prime for $n \notin \{-1, 0, 1\}$. $\square$

3.7. Total picture. As a consequence, $\mathit{Spec}\,\Omega$ can be determined as

$$\begin{aligned}\mathit{Spec}\,\Omega &= (\{(0)\}\cup\{\mathscr{I}(0)\})\ \cup\ \{\mathfrak{m}_q\}\\ &\quad \cup\ (\{\mathfrak{p}_p \mid p\in\mathbb{Z}\text{ is prime, } p\neq q\}\cup\{\mathfrak{m}_p\mid p\in\mathbb{Z}\text{ is prime, } p\neq q\}).\end{aligned}$$

Inclusions are

$$\begin{array}{ccccc} (0) & \subsetneq & \mathscr{I}_{(0)} & \subsetneq & \mathfrak{m}_q \\ \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} & & \\ \mathfrak{p}_p & \subsetneq & \mathfrak{m}_p & & {\scriptstyle (p\neq q)}\ . \end{array}$$

Especially the dimension of $\mathit{Spec}\,\Omega$ is 2.

$\mathfrak{m}_q$ and $\mathfrak{m}_p$'s are the closed points, and (0) is the generic point in $\mathit{Spec}\,\Omega$. If we represent the points in $\mathit{Spec}\,\Omega$ by their closures, $\mathit{Spec}\,\Omega$ with fibration F can be depicted as follows. It can be also easily seen that F is continuous.

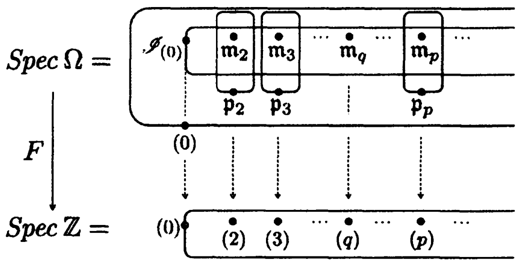

FIGURE 1. $\mathit{Spec}\,\Omega$ for $G=\mathbb{Z}/q\mathbb{Z}$

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